Unit 10 - Derivatives

Unit 10 Day #1 - Intro to Derivatives

Objective - You will (1) Write the equation of a line given: a) m and a pt on the line
b) 2 pts on the line
(2) Evaluate a) $f(x+h)-f(x)$
b) $f(x+h)-f(x)/h$

Writing the equation of a Line

Slope-Intercept Form: $y=mx+b$

Point-Slope Form: $y-y_1=m(x-x_1)$

Recall: slope: $\frac{\Delta y}{\Delta x}=\frac{y_2-y_1}{x_2-x_1}$

Examples:

1) Find the equation of a line given $m=-3$ and the point $(-2,-5)$ on the line.

$$y-y_1=m(x-x_1)$$
$$y-(-5)=-3(x-(-2))$$
$$y+5=-3(x+2)$$
$$y+5=-3x-6$$
$$-5 \qquad -5$$
$$\boxed{y=-3x-11}$$

2) Find the equation of a line passing through $(1,6)$ and $(-4,3)$.

$(1,6)$ $(-4,3)$

$m=\frac{y_2-y_1}{x_2-x_1}=\frac{3-6}{-4-1}$

$=\frac{-3}{-5}$

$\boxed{m=\frac{3}{5}}$

$$y-y_1=m(x-x_1)$$
$$y-6=\frac{3}{5}(x-1)$$
$$y-6=\frac{3}{5}x-\frac{3}{5}$$
$$+6 \qquad +6$$
$$\boxed{y=\frac{3}{5}x+\frac{27}{5}}$$

$\frac{-3}{5}+\frac{30}{5}$

Evaluating a Function

Substitute the given # for x.

ex Given: $f(x) = x+1$

Evaluate: a) $f(3) = 3+1 = \boxed{4}$

b) $f(★) = \boxed{★ + 1}$

c) $f(\square + \triangle) = \boxed{(\square + \triangle) + 1}$

Example: Given $f(x) = 3x^2 + 2x$

Evaluate: a) $f(x+h) = 3(x+h)^2 + 2(x+h)$

$= 3(x^2 + 2xh + h^2) + 2x + 2h$

$\boxed{= 3x^2 + 6xh + 3h^2 + 2x + 2h}$

b) $f(x+h) - f(x)$

$= 3x^2 + 6xh + 3h^2 + 2x + 2h - (3x^2 + 2x)$

$= \cancel{3x^2} + 6xh + 3h^2 \cancel{+2x} + 2h \cancel{-3x^2} \cancel{-2x}$

$\boxed{= 6xh + 3h^2 + 2h}$

c) $\dfrac{f(x+h) - f(x)}{h} = \dfrac{6xh + 3h^2 + 2h}{h}$

$\boxed{= 6x + 3h + 2}$

Homework

1) Write the equation of a line that has a slope of -3 and goes through the point $(5,2)$

2) Write the equation of a line that passes through the points $(2,1)$ and $(4,-4)$.

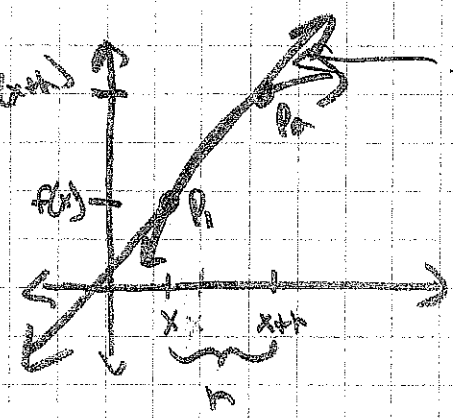
3) Given $f(x) = 3x^2 - 6x + 5$

Evaluate a) $f(x+h)$ b) $f(x+h) - f(x)$ c) $\dfrac{f(x+h) - f(x)}{h}$

Unit 10 Day #2 – The Derivative

Objective – You will use the slope of the secant line to find the equation of the tangent line @ a pt on the curve.

Derivative – the slope of a line tangent to a curve at any given point.
– the rate of change.



secant line

Secant line: $m_{sec} = \dfrac{\Delta y}{\Delta x} = \dfrac{f(x+h) - f(x)}{(x+h) - x}$

$= \dfrac{f(x+h) - f(x)}{h}$

$\boxed{m_{tan} = \lim_{h \to 0} m_{sec}}$

Example: Find the slope of the parabola $f(x) = x^2 + 2x$ at $x = 2$.
– write the equation for the tangent to the parabola @ this point (need m and point on line)

$m_{sec} = \dfrac{f(x+h) - f(x)}{h} = \dfrac{\overbrace{(x+h)^2 + 2(x+h)}^{f(x+h)} - \overbrace{(x^2+2x)}^{f(x)}}{h}$

$= \dfrac{\cancel{x^2} + 2xh + h^2 \cancel{+2x} + 2h \cancel{- x^2} \cancel{- 2x}}{h}$

$= \dfrac{2xh + h^2 + 2h}{h} = \boxed{2x + h + 2}$ } Slope of any secant line at any pt on curve

$m_{tan} = \lim_{h \to 0} m_{sec} = \lim_{h \to 0} 2x + h + 2 = 2x + 2$ } m of any tangent line @ any pt on curve.

m tan @ x = 2: $2x + 2$
$= 2(2) + 2$
$= 6$

Pt on curve: $x = 2$
$y = x^2 + 2x$
$y = 2^2 + 2(2)$
$y = 8$

Tan line to $y = x^2 + 2x$ @ $(2, 8)$
$y - y_1 = m(x - x_1)$
$y - 8 = 6(x - 2)$
$y - 8 = 6x - 12$
$+8 \qquad +8$
$\boxed{y = 6x - 4}$

Supplement Example

Find the derivative of $f(x) = 2x^2+3$ at $x=3$, then find the slope of the curve

$$m \text{ of sec} = \frac{f(x+h)-f(x)}{h} = \frac{2(x+h)^2+3-(2x^2+3)}{h}$$

$$= \frac{2(x^2+2xh+h^2)+3-2x^2-3}{h}$$

$$= \frac{\cancel{2x^2}+4xh+2h^2\cancel{+3}\cancel{-2x^2}\cancel{-3}}{h}$$

$$= \frac{4xh+2h^2}{h} = \boxed{4x+2h}$$

$$m \text{ of tan} = \lim_{h\to 0} 4x+2h = 4x+2(0) = \boxed{4x}$$

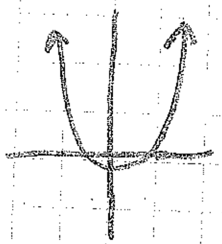
$$m \text{ of tan @ } x=3 = 4(3) = \boxed{12}$$

$y = 2x^2+3$
$y = 2(3)^2+3$
$y = 21$

$m=12 \quad (3,21)$
$y-y_1 = m(x-x_1)$
$y-21 = 12(x-3)$
$y-21 = 12x-36$
$+21 \qquad +21$
$\boxed{y=12x+15}$

2) Find the slope of the line tangent to the curve $f(x) = x^2 - 2x + 3$ at $\boxed{x = -1}$.

$$m_{sec} = \frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 - 2(x+h) + 3 - (x^2 - 2x + 3)}{h}$$



$$= \frac{\cancel{x^2} + 2xh + h^2 \cancel{-2x} - 2h \cancel{+3} \cancel{-x^2} \cancel{+2x} \cancel{-3}}{h}$$

$$= \frac{2xh + h^2 - 2h}{h} = \boxed{2x + h - 2}$$

$$m_{tan} = \lim_{h \to 0} m_{sec} = 2x + h - 2 = 2x + 0 - 2 = \boxed{2x - 2}$$

m tan @ $x = -1$; $2(-1) - 2 = \boxed{-4}$

b) Write the equation of the line tangent to the given curve at the given point

Pt on curve : $x = -1$
$y = (-1)^2 - 2(-1) + 3$
$y = 6$
$(-1, 6)$

$y - y_1 = m(x - x_1)$
$y - 6 = -4(x + (+1))$
$y - 6 = -4x - 4$
$+6 \qquad +6$
$\boxed{y = -4x + 2}$

Homework: 1) Write the equation of a line tangent to $f(x) = x^2 - 1$ at $(2, 3)$

2) Write the equation of a line tangent to $f(x) = x^2 + 7x$ at $x = 0$

Unit 10 Day #3 – Techniques of Differentiation

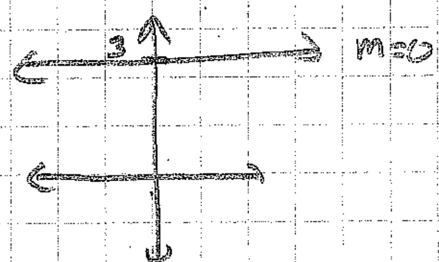
Objective – You will calculate derivatives more efficiently.

Techniques of Differentiation

Derivative Notation: $\frac{dy}{dx}$, $f'(x)$, y'

Rule #1: The derivative of a constant is 0.

Why? If $f(x)=3$ (horizontal line), the slope (derivative) is 0.



Rule 2 + 3: The Power Rule and the Constant Multiple Rule

$$\frac{d}{dx}\left[cx^n\right] = c\,n\,x^{n-1}$$

Examples:

$f(x)$		$f'(x)$
$3x^2$	$\longrightarrow$	$6x$
$4x^3$	$\longrightarrow$	$12x^2$
$7x$	$\longrightarrow$	7
2	$\longrightarrow$	0
$6x^{-2}$	$\longrightarrow$	$-12x^{-3}$
$\frac{1}{x^4} = x^{-4}$	$\longrightarrow$	$-4x^{-5}$

* Show $\frac{d}{dx}\sqrt{x} = x^{\frac{1}{2}}$

$= \frac{1}{2}x^{-\frac{1}{2}}$

$= \frac{1}{2\sqrt{x}}$

The Sum and Difference Rule

- Take the derivative of each term.

1) $y = x^3 + 3x^2 - 5x + 1$

$y' = 3x^2 + 6x - 5$

2) $y = \frac{7x^2}{3} - 5$

$y' = \frac{14x}{3}$

3) $f(x) = \frac{x^3 + 4}{6}$

$f'(x) = \frac{3x^2}{6} = \frac{1}{2}x^2$

4) $f(x) = \frac{1}{5}(x^7 - 4)$

$f'(x) = \frac{7}{5}x^6$

5) $g(x) = -4x^{-3} + 4\sqrt{x}$ $\quad 4(x)^{1/2}$

$g'(x) = 12x^{-4} + 2x^{-1/2}$

Second & Higher Order Derivatives

- Taking the derivative of a derivative

Find the third derivative of:

1) $y = 2x^3 + 3x^2 - x$

$y' = 6x^2 + 6x - 1$

$y'' = 12x + 6$

$y''' = 12$

2) $f(x) = -3x^{-2} - x^2$

$f'(x) = 6x^{-3} - 2x$

$f''(x) = -18x^{-4} - 2$

$f'''(x) = 72x^{-5}$

HW: Pg 198 (#1-4, 7, 8, 11, 13, 14)

Unit 10 Day #4 – Tangent Lines to a Curve

Objective – You will find the slope and equation of a line tangent to a curve @ a given point.

We are now going to write equations of tangent lines using the quick way of calculating derivatives (slopes)!

Recall – Formula for the equation of a line:

$$y - y_1 = m(x - x_1)$$

Examples: Find the equation of the line tangent to the curve at the given point.

1) $y = x^2 - 3$ at $(3, 6)$

* You need m and a pt on the line, so.... $m \tan = \frac{dy}{dx}$

$m \tan = \frac{dy}{dx} = 2x$

$m \tan$ @ $x = 3 = 2(3) = 6$

$(3, 6)$ $m = 6$

$y - y_1 = m(x - x_1)$

$y - 6 = 6(x - 3)$

$y - 6 = 6x - 18$

$+6 \quad\quad +6$

$\boxed{y = 6x - 12}$

2) $y = x^2 - 5x + 6$ at $x = 4$

$m \tan = \frac{dy}{dx} = 2x - 5$

$m \tan$ @ $x = 4$: $2(4) - 5 = \boxed{3}$

If $x = 4$, $y = (4)^2 - 5(4) + 6$

$y = 2$

$(4, 2)$ $m = 3$

$y - y_1 = m(x - x_1)$

$y - 2 = 3(x - 4)$

$y - 2 = 3x - 12$

$+2 \quad\quad +2$

$\boxed{y = 3x - 10}$

3) Find the slope of the line tangent to $y = x^2 - 3x + 2$ at $x = 1$ using the slope of the secant. (long way)

$$m \text{ of sec} = \frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 - 3(x+h) + 2 - (x^2 - 3x + 2)}{h}$$

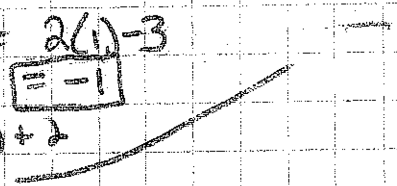
$$= \frac{\cancel{x^2} + 2xh + h^2 \cancel{- 3x} - 3h \cancel{+2} \cancel{- x^2} \cancel{+ 3x} \cancel{- 2}}{h}$$

$$= \frac{2xh + h^2 - 3h}{h} = \boxed{2x + h - 3}$$

$$m \text{ of tan} = \lim_{h \to 0} 2x + h - 3 = 2x + 0 - 3$$
$$\boxed{= 2x - 3}$$

$$m \text{ of tan } @ \ x = 1 = 2(1) - 3$$
$$\boxed{= -1}$$

$y = (1)^2 - 3(1) + 2$
$y = 0$



$(1, 0) \quad m = -1$
$y - y_1 = m(x - x_1)$
$y - 0 = -1(x - 1)$
$\boxed{y = -x + 1}$

HW: 1) Find the equation of the line tangent to $y = 0.5x^2 + x - 1$ @ $(4, 11)$

2) Write the equation of the line tangent to $y = -x^2 - x + 2$ @ $x = 0.5$

3) Using slope of the secant (LONG WAY) find the equation the line tangent to $f(x) = 2x^2 + 3$ @ $x = 2$

Unit 10 Day #5 – Product and Quotient Rules

Objective – You will use derivative rules that involve a product or a quotient.

Product Rule

The derivative of $f \cdot g = g \cdot f' + f \cdot g'$

ex: Find $\frac{dy}{dx}$ of $y = \underset{f}{\tfrac{1}{2}}\underset{g}{(x^4+7)}$

$$\frac{dy}{dx} = (x^4+7)(0) + \tfrac{1}{2}(4x^3)$$

$$= 0 + 2x^3 = \boxed{2x^3}$$

example Find $\frac{dy}{dx}$ of $y = \overset{f}{(x^2+1)}\overset{g}{(x^3+3)}$

$$\frac{dy}{dx} = (x^3+3)(2x) + (x^2+1)(3x^2)$$

$$\frac{dy}{dx} = 2x^4 + 6x + 3x^4 + 3x^2$$

$$= 5x^4 + 3x^2 + 6x$$

Quotient Rule

The derivative of $\frac{f}{g} = \frac{g \cdot f' - f \cdot g'}{g^2}$

Ex: Find $\frac{dy}{dx}$ of $y = \frac{x^2+1 \;^{f}}{5 \;_{g}}$

$$\frac{dy}{dx} = \frac{5(2x) - (x^2+1)(0)}{5^2} = \frac{10x - 0}{25}$$

$$= \frac{10x}{25}$$

$$\boxed{= \frac{2x}{5}}$$

Example:

Find $\frac{dy}{dx}$ of $y = \frac{x^2-1}{x^2+1}\begin{matrix} f \\ g \end{matrix}$

$$\frac{dy}{dx} = \frac{(x^2+1)(2x) - (x^2-1)(2x)}{(x^2+1)^2}$$

$$= \frac{2x^3 + 2x - (2x^3 - 2x)}{(x^2+1)(x^2+1)}$$

$$= \frac{\cancel{2x^3} + 2x - \cancel{2x^3} + 2x}{x^4 + 2x^2 + 1} = \frac{4x}{x^4+2x^2+1}$$

HW: Pg 198 # 15, 16, 25, 26

Unit 10 Day #6 - Derivatives in Trigonometric Functions

<u>Objective</u> - You will use the rules for differentiating the 6 basic trig functions.

<u>Trig Function</u>	<u>Derivative</u>
$\sin\theta$	$\cos\theta$
$\cos\theta$	$-\sin\theta$
$\tan\theta$	$\sec^2\theta$
$\cot\theta$	$-\csc^2\theta$
$\sec\theta$	$\sec\theta\tan\theta$
$\csc\theta$	$-\csc\theta\cot\theta$

<u>Examples</u>

1) $f(x) = 3\tan x$

$f'(x) = 3\cdot\sec^2 x$

2) $f(x) = 2\cos x - 3\sin x$

$f'(x) = 2(-\sin x) - 3(\cos x)$
$= -2\sin x - 3\cos x$

3) $f(x) = \sin x \cdot \cos x \quad = gf' + fg'$

$f = \sin x \qquad g = \cos x$
$f' = \cos x \qquad g' = -\sin x$

$= \cos x(\cos x) + \sin x(-\sin x)$

$= \boxed{\cos^2 x - \sin^2 x} = \boxed{\cos 2x}$

4) $f(x) = \dfrac{\csc x}{\tan x} = \dfrac{g\cdot f' - f\cdot g'}{g^2} = \dfrac{\tan x(-\csc x\cot x) - \csc x(\sec^2 x)}{\tan^2 x}$

$f = \csc x$
$f' = -\csc x\cot x$

$g = \tan x$

$g' = \sec^2 x$

$= \dfrac{\left(\dfrac{\sin x}{\cos x}\cdot\dfrac{-1}{\sin x}\cdot\dfrac{\cos x}{\sin x}\right) - \dfrac{1}{\sin x}\left(\dfrac{1}{\cos^2 x}\right)}{\tan^2 x}$

$= \dfrac{\dfrac{-1}{\sin x} - \dfrac{1}{\sin x\cos^2 x}}{\tan^2 x}$

5) $f(x) = \underbrace{x^4 \csc x} - 4\sec x$

$f = x^4$ $\quad$ $g = \csc x$

$f' = 4x^3$ $\quad$ $g' = -\csc x \cot x$

$f'(x) = g \cdot f' + f \cdot g' - 4\sec x \tan x$

$f'(x) = \csc x(4x^3) + (x^4)(-\csc x \cot x) - 4\sec x \tan x$

$$\boxed{f'(x) = 4x^3 \csc x - x^4 \csc x \cot x - 4\sec x \tan x}$$

Homework: Pg 203 # 3-5, 9, 16

Unit 10 Day #7 - The Power Rule and Chain Rule

Objective - You will differentiate composite functions using the power rule and chain rule.

The Power Rule

Used when finding a derivative raised to a power.

$$y = [f(x)]^n$$

$$y' = n \cdot [f(x)]^{n-1} \cdot f'(x)$$

Examples:

1) $y = (2x+1)^{-3}$

$y' = -3(2x+1)^{-4} \cdot \frac{d}{dx}[2x+1]$

~~$y' = -3(2x+1)^{-4} \cdot (2x)$~~

$\boxed{y' = -6x(2x+1)^{-4}}$ or $\boxed{\frac{-6x}{(2x+1)^4}}$

2) $y = \sqrt{4x^2+3x-1} = y = (4x^2+3x-1)^{1/2}$

$y' = \frac{1}{2}(4x^2+3x-1)^{-1/2} \frac{d}{dx}[4x^2+3x-1]$

$y' = \frac{1}{2}(4x^2+3x-1)^{-1/2} \cdot 8x+3$

$\boxed{y' = 4x + 3/2\,(4x^2+3x-1)^{-1/2}}$

3) $y = \frac{15}{(x^2+3x)^4} = y = 5 \cdot (x^2+3x)^{-4}$

$y' = -20(x^2+3x)^{-5} \cdot \frac{d}{dx}[2x+3]$

$y' = \frac{-20(2x+3)}{(x^2+3x)^5}$

4) $y = \sin^5 x$

$y = (\sin x)^5$

$y' = 5(\sin x)^4 \cdot \frac{d}{dx}(\sin x)$

$y' = 5 \sin x^4 \cdot \cos x$

$y' = 5 \sin x^4 \cos x$

The Chain Rule

Used when taking the derivative of a composition.

$$y = f[g(x)] \qquad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$y' = f'[g(x)] \cdot g'(x)$$

Examples:

1) $y = \sin(x^2-4)$

$y' = \cos(x^2-4) \cdot (2x)$

$\boxed{y' = 2x\cos(x^2-4)}$

2) $y = \sin(\sin x)$

$y' = \cos(\sin x) \cdot (\cos x)$ *

$y' = \cos x \cdot \cos(\sin x)$

$y = \cos(\sin x) \cdot \frac{d}{dx}[\sin x)$

$\boxed{y = \cos(\sin x) \cdot \cos x}$

3) $y = \tan(3\sqrt{x})$

$\frac{dy}{dx} = \sec^2(3\sqrt{x}) \cdot \frac{d}{dx}\left[3x^{1/2}\right]$

$= \sec^2(3\sqrt{x}) \cdot \frac{3}{2}x^{-1/2}$

$\boxed{= \sec^2(3\sqrt{x}) \cdot \frac{3}{2\sqrt{x}}}$

$y = x^{1/2}$

$y' = \frac{1}{2}x^{-1/2}$

$\boxed{y' = \frac{1}{2\sqrt{x}}}$

4) $y = \sec(5x^2+3)$

$\frac{dy}{dx} = \sec(5x^2+3)\tan(5x^2+3) \cdot \frac{d}{dx}\left[5x^2+3\right]$

$\frac{dy}{dx} = \sec(5x^2+3)\tan(5x^2+3) \cdot 10x$

$= 10x\sec(5x^2+3)\tan(5x^2+3)$

5) $y = (5x+8)^{13}(x^3+7x)^{12}$ $\quad y' = g\cdot f' + f\cdot g'$

$$\frac{dy}{dx} = (x^3+7x)^{12}\cdot 13(5x+8)^{12}\frac{d}{dx}[5x+8] + (5x+8)^{13}\cdot 12(x^3+7x)^{11}\frac{d}{dx}[x^3+7x]$$

$$\frac{dy}{dx} = (x^3+7x)^{12}\cdot 13(5x+8)^{12}\cdot(5) + (5x+8)^{13}\cdot 12(x^3+7x)^{11}\cdot(3x^2+7)$$

$$\frac{dy}{dx} = 65(\underbrace{x^3+7x}_{a})^{12}(\underbrace{5x+8}_{b})^{12} + 12(\underbrace{5x+8}_{b})^{13}\cdot(\underbrace{x^3+7x}_{a})^{11}\cdot(3x^2+7)$$

$$\frac{dy}{dx} = 65a^{12}b^{12} + 12a^{11}b^{13}\cdot(3x^2+7)$$

$$\frac{dy}{dx} = a^{11}b^{12}\left(65a + 12b\cdot(3x^2+7)\right)$$

$$\boxed{\frac{dy}{dx} = (x^3+7x)^{11}(5x+8)^{12}\left(65(x^3+7x) + 12(5x+8)(3x^2+7)\right)}$$

HW:

7) $f(x) = (x^3+2x)^{37}$

11) $f(x) = \dfrac{4}{(3x^2-2x+1)^3}$

12) $f(x) = \sqrt{x^2-2x+5}$

14) $f(x) = \sin^3 x$

15) $f(x) = \sin(x^3)$

19) $f(x) = \sin\left(\dfrac{1}{x^2}\right)$

31) $y = \cos(\cos x)$